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Green's dyadic approach of the self-stress on a dielectric–diamagnetic cylinder with non-uniform speed of light

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Abstract

We present a Green's dyadic formulation to calculate the Casimir energy for a dielectric–diamagnetic cylinder with the speed of light differing inside and outside. Although the result is in general divergent, special cases are meaningful. It is pointed out how the self-stress on a purely dielectric cylinder vanishes through second order in the deviation of the permittivity from its vacuum value, in agreement with the result calculated from the sum of van der Waals forces.

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1. Formulation of the Green's dyadic approach

The electromagnetic Green's dyadic functions [1] have been successfully used on many occasions (for an extensive view see [2] and references therein) and can be applied to very complicated geometries. Their use is critical in this calculation [3]. This approach helps us to compute the vacuum expectation value of the fields rigorously; we show that the approach is both illuminating of the physics and unambiguous.

1.1. Green's dyadic equations; formalism

In a medium of constant electric permittivity ε' and magnetic permeability μ', we insert an infinitely long cylinder of radius a with permittivity ε and permeability μ. The product of these parameters is different from that of the outside parameters. There are no real charges of

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any kind present in the problem, $\rho = \mathbf{J} = 0$, and since we work at a fixed frequency we can Fourier transform the electric and magnetic fields,

$$\mathbf{E}(\mathbf{r}, t) = \int_{\infty}^{\infty} \frac{\mathrm{d}\omega}{2\pi} \mathbf{E}(\mathbf{r}, \omega)\, \mathrm{e}^{-\mathrm{i}\omega t}, \qquad \mathbf{B}(\mathbf{r}, t) = \int_{\infty}^{\infty} \frac{\mathrm{d}\omega}{2\pi} \mathbf{B}(\mathbf{r}, \omega)\, \mathrm{e}^{-\mathrm{i}\omega t}, \tag{1}$$

and the corresponding Maxwell's equations are

$$\nabla \times \mathbf{E} = \mathrm{i}\omega\mu\mathbf{H}, \qquad \nabla \cdot \mathbf{D} = 0, \tag{2a}$$

$$\nabla \times \mathbf{H} = -\mathrm{i}\omega\varepsilon\mathbf{E}, \qquad \nabla \cdot \mathbf{B} = 0. \tag{2b}$$

In order to write the Green's dyadic equations, we introduce a polarization source $\mathbf{P}$. The first equation in (2b) and the second one in (2a) then get changed to

$$\nabla \times \mathbf{H} = -\mathrm{i}\omega\varepsilon\mathbf{E} - \mathrm{i}\omega\mathbf{P}, \qquad \nabla \cdot \mathbf{D} = -\nabla \cdot \mathbf{P}. \tag{3}$$

The linear relation of polarization source with the electric field defines the Green's dyadic as

$$\mathbf{E}(x) = \int (\mathrm{d}x')\boldsymbol{\Gamma}(x, x') \cdot \mathbf{P}(x'). \tag{4}$$

Since the response is translationally invariant in time, we work with the Fourier transform of the dyadic at a given frequency ω. We can then, by simple substitution, write the dyadic Maxwell's equations in a medium characterized by a dielectric constant ε and a permeability μ:[2]

$$\nabla \times \boldsymbol{\Gamma}' - \mathrm{i}\omega\mu(\omega)\boldsymbol{\Phi} = \frac{1}{\varepsilon(\omega)}\nabla \times \mathbf{1}, \qquad \nabla \cdot \boldsymbol{\Phi} = \mathbf{0}, \tag{5a}$$

$$-\nabla \times \boldsymbol{\Phi} - \mathrm{i}\omega\varepsilon(\omega)\boldsymbol{\Gamma}' = \mathbf{0}, \qquad \nabla \cdot \boldsymbol{\Gamma}' = \mathbf{0}, \tag{5b}$$

and where the unit dyadic $\mathbf{1}$ includes a three-dimensional δ function, $\mathbf{1} = \mathbf{1}\delta(\mathbf{r} - r')$. Quantum mechanically, these Green's dyadics give the one-loop vacuum expectation values of the product of fields at a given frequency ω,

$$\langle \mathbf{E}(\mathbf{r})\mathbf{E}(\mathbf{r}')\rangle = \frac{\hbar}{\mathrm{i}}\boldsymbol{\Gamma}(\mathbf{r}, \mathbf{r}'), \qquad \langle \mathbf{H}(\mathbf{r})\mathbf{H}(\mathbf{r}')\rangle = -\frac{\hbar}{\mathrm{i}}\frac{1}{\omega^2\mu^2}\overrightarrow{\nabla} \times \boldsymbol{\Gamma}(\mathbf{r}, \mathbf{r}') \times \overleftarrow{\nabla}'. \tag{6}$$

Thus, from the knowledge of the classical Green's dyadics, we can calculate the vacuum energy or stress.

Since the TE and TM modes do not separate, we cannot use the general waveguide decomposition of modes into those of TE and TM types[3]. However, we can introduce the appropriate partial wave decomposition for a cylinder, in terms of cylindrical coordinates (r, θ, z)[4]:

$$\boldsymbol{\Gamma}'(\mathbf{r}, \mathbf{r}'; \omega) = \sum_{m=-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{\mathrm{d}k}{2\pi} \bigg\{ (\nabla \times \hat{\mathbf{z}}) f_m(r; k, \omega)\chi_{mk}(\theta, z) + \frac{\mathrm{i}}{\omega\varepsilon}\nabla \times (\nabla \times \hat{\mathbf{z}}) g_m(r; k, \omega)\chi_{mk}(\theta, z) \bigg\}, \tag{7a}$$

[2] In order to have divergenceless Green dyadics, we redefine the electric Green's dyadic in the following way: $\boldsymbol{\Gamma}'(\mathbf{r}, \mathbf{r}', \omega) = \boldsymbol{\Gamma}(\mathbf{r}, \mathbf{r}', \omega) + \frac{1}{\varepsilon(\omega)}\delta(\mathbf{r} - \mathbf{r}')$ and $\boldsymbol{\Phi}$ is the magnetic dyadic.

[3] For example as given in [4]. However, this is here impossible because the TE and TM modes do not separate. See [5].

[4] A slight modification of that given for a conducting cylindrical shell [6].

$$\boldsymbol{\Phi}(\mathbf{r},\mathbf{r}';\omega)=\sum_{m=-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{\mathrm{d}k}{2\pi}\bigg\{(\nabla\times\hat{\mathbf{z}})\tilde{g}_m(r;k,\omega)\chi_{mk}(\theta,z)$$
$$-\frac{\mathrm{i}\varepsilon}{\omega\mu}\nabla\times(\nabla\times\hat{\mathbf{z}})\tilde{f}_m(r;k,\omega)\chi_{mk}(\theta,z)\bigg\},\qquad(7b)$$

where the cylindrical harmonics are $\chi(\theta,z)=\frac{1}{\sqrt{2\pi}}\,\mathrm{e}^{\mathrm{i}m\theta}\,\mathrm{e}^{\mathrm{i}kz}$ and the dependence of f_m, etc on $\mathbf{r}'$ is implicit. Note that these are vectors in the second tensor index. Because of the presence of these harmonics, we have

$$\nabla\times\hat{\mathbf{z}}\to\hat{\mathbf{r}}\frac{\mathrm{i}m}{r}-\hat{\boldsymbol{\theta}}\frac{\partial}{\partial r}\equiv\boldsymbol{\mathcal{M}}\quad\text{and}\quad\nabla\times(\nabla\times\hat{\mathbf{z}})\to\hat{\mathbf{r}}\mathrm{i}k\frac{\partial}{\partial r}-\hat{\boldsymbol{\theta}}\frac{mk}{r}-\hat{\mathbf{z}}d_m\equiv\boldsymbol{\mathcal{N}},\qquad(8)$$

in terms of the cylinder operator $d_m=\frac{1}{r}\frac{\partial}{\partial r}r\frac{\partial}{\partial r}-\frac{m^2}{r^2}$. It is trivial to see that the divergence of (7a) and (7b) is zero, satisfying immediately the two dyadic Maxwell's equations. Now, if we use the Maxwell equation (5b), we conclude[5]

$$\tilde{g}_m=g_m\qquad\text{and}\qquad(d_m-k^2)\tilde{f}_m=-\omega^2\mu f_m.\qquad(9)$$

More elaborate work is needed to obtain a condition from the other Maxwell equation (5a). Using the above we can write (5a) as

$$\sum_m\int_{-\infty}^{\infty}\frac{\mathrm{d}k}{2\pi}\bigg\{-\boldsymbol{\mathcal{M}}\frac{(d_m-k^2)}{\omega^2\mu}\tilde{f}_m-\frac{\mathrm{i}}{\omega\varepsilon}(d_m-k^2)\boldsymbol{\mathcal{N}}g_m\bigg\}\chi_{mk}(\theta,z)$$
$$=\sum_m\int_{-\infty}^{\infty}\frac{\mathrm{d}k}{2\pi}\{\mathrm{i}\omega\mu\boldsymbol{\mathcal{N}}g_m+\varepsilon\boldsymbol{\mathcal{M}}\tilde{f}_m\}\chi_{mk}(\theta,z)+\frac{1}{\varepsilon}\nabla\times\mathbf{1}.\qquad(10)$$

If we multiply the above by the expression $\int_0^{2\pi}\int_{-\infty}^{\infty}\mathrm{d}\theta\,\mathrm{d}z\,\chi^*_{m'k'}(\theta,z)$ and apply $\int_0^{2\pi}\int_{-\infty}^{\infty}\mathrm{d}\theta\,\mathrm{d}z\,\chi^*_{m'k'}(\theta,z)\chi_{mk}(\theta,z)=2\pi\delta(k-k')\delta_{mm'}$, we find

$$-\frac{1}{\omega^2\mu}\boldsymbol{\mathcal{N}}(d_m-k^2+\omega^2\mu\varepsilon)\tilde{f}_m-\frac{\mathrm{i}}{\omega\varepsilon}\boldsymbol{\mathcal{M}}(d_m-k^2+\omega^2\mu\varepsilon)g_m$$
$$=\frac{1}{\varepsilon}\int_0^{2\pi}\int_{-\infty}^{\infty}\mathrm{d}\theta\,\mathrm{d}z\,\chi^*_{mk}(\theta,z)(\nabla\times\mathbf{1})\frac{1}{r}\delta(r-r')\delta(\theta-\theta')\delta(z-z'),\qquad(11)$$

where the delta functions are now made explicit. By dotting this expression with $\hat{\mathbf{z}}$, we note that $\hat{\mathbf{z}}\cdot\boldsymbol{\mathcal{M}}=0$ and $\hat{\mathbf{z}}\cdot\boldsymbol{\mathcal{N}}=-d_m$ and after a little manipulation we arrive at the fourth-order differential equation:

$$d_m\mathcal{D}_m\tilde{\mathbf{f}}_m(r;r',\theta',z')=\frac{\omega^2\mu}{\varepsilon}\boldsymbol{\mathcal{M}}'^*\frac{1}{r}\delta(r-r')\chi^*_{mk}(\theta',z').\qquad(12)$$

If we now dot it with $(\nabla\times\hat{\mathbf{z}})$, we learn that a similar equation holds for g_m:

$$d_m\mathcal{D}_m\mathbf{g}_m(r;r',\theta',z')=-\mathrm{i}\omega\boldsymbol{\mathcal{N}}'^*\frac{1}{r}\delta(r-r')\chi^*_{mk}(\theta',z'),\qquad(13)$$

where we have made the second, previously suppressed, position arguments explicit and the prime on the differential operator signifies action on the second primed argument[6].

To solve these equations, we separate variables in the second argument,

$$\tilde{\mathbf{f}}_m(r,\mathbf{r}')=\left[\boldsymbol{\mathcal{M}}'^*F_m(r,r';k,\omega)+\frac{1}{\omega}\boldsymbol{\mathcal{N}}'^*\tilde{F}_m(r,r';k,\omega)\right]\chi^*_{mk}(\theta',z'),\qquad(14a)$$

[5] The ambiguity in solving these equations is absorbed in the definition of subsequent constants of integration.
[6] The Bessel operator appears, $\mathcal{D}_m=d_m+\lambda^2$, $\lambda^2=\omega^2\varepsilon\mu-k^2$.

$$\mathbf{g}_m(r, \mathbf{r}') = \left[-\frac{\mathrm{i}}{\omega}\boldsymbol{\mathcal{N}}'^* G_m(r, r'; k, \omega) - \mathrm{i}\boldsymbol{\mathcal{M}}'^* \tilde{G}_m(r, r'; k, \omega)\right]\chi^*_{mk}(\theta', z'), \tag{14b}$$

where we have introduced the two scalar Green's functions F_m, G_m satisfying

$$d_m\mathcal{D}_m F_m(r, r') = \frac{\omega^2\mu}{\varepsilon}\frac{1}{r}\delta(r - r') \qquad \text{and} \qquad d_m\mathcal{D}_m G_m(r, r') = \omega^2\frac{1}{r}\delta(r - r'), \tag{15}$$

while $\tilde{F}_m$ and $\tilde{G}_m$ are annihilated by the operator $d_m\mathcal{D}_m$,

$$d_m\mathcal{D}_m\tilde{F}(r, r') = d_m\mathcal{D}_m\tilde{G}(r, r') = 0. \tag{16}$$

1.2. Green's dyadic solutions

The Green's dyadics now have the form

$$\begin{aligned}\boldsymbol{\Gamma}'(\mathbf{r}, \mathbf{r}'; \omega) = \sum_{m=-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{\mathrm{d}k}{2\pi}\bigg\{&\boldsymbol{\mathcal{M}}\boldsymbol{\mathcal{M}}'^*\left(-\frac{d_m - k^2}{\omega^2\mu}\right)F_m(r, r') + \boldsymbol{\mathcal{N}}\boldsymbol{\mathcal{N}}'^*\frac{1}{\omega^2\varepsilon}G_m(r, r')\\ &+\frac{1}{\omega}\boldsymbol{\mathcal{M}}\boldsymbol{\mathcal{N}}'^*\left(-\frac{d_m - k^2}{\omega^2\mu}\right)\tilde{F}_m(r, r') + \frac{1}{\omega\varepsilon}\boldsymbol{\mathcal{N}}\boldsymbol{\mathcal{M}}'^*\tilde{G}_m(r, r')\bigg\}\chi_{mk}(\theta, z)\chi^*_{mk}(\theta', z'),\end{aligned} \tag{17a}$$

$$\begin{aligned}\boldsymbol{\Phi}(\mathbf{r}, \mathbf{r}'; \omega) = \sum_{m=-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{\mathrm{d}k}{2\pi}\bigg\{&-\frac{\mathrm{i}}{\omega}\boldsymbol{\mathcal{M}}\boldsymbol{\mathcal{N}}'^* G_m(r, r') - \frac{\mathrm{i}\varepsilon}{\omega\mu}\boldsymbol{\mathcal{N}}\boldsymbol{\mathcal{M}}'^* F_m(r, r')\\ &-\mathrm{i}\boldsymbol{\mathcal{M}}\boldsymbol{\mathcal{M}}'^*\tilde{G}_m(r, r') - \frac{\mathrm{i}\varepsilon}{\omega^2\mu}\boldsymbol{\mathcal{N}}\boldsymbol{\mathcal{N}}'^*\tilde{F}_m(r, r')\bigg\}\chi_{mk}(\theta, z)\chi^*_{mk}(\theta', z').\end{aligned} \tag{17b}$$

In the following, we will apply these equations to a dielectric–diamagnetic cylinder of radius a, where the interior of the cylinder is characterized by a permittivity ε and permeability μ, while the outside is vacuum, so $\varepsilon = \mu = 1$ there. Let us consider the case that the source point is outside, $r' > a$. If the field point is also outside, $r, r' > a$, the scalar Green's functions $F'_m, G'_m, \tilde{F}', \tilde{G}'$ that make up the above Green's dyadics (we designate with primes the outside scalar Green's functions or constants) obey the differential equations (15) and (16) with $\varepsilon = \mu = 1$. The solutions to these equations are[7]

$$\begin{aligned}F'_m(r, r') = {}&\frac{\omega^2}{\lambda'^2}\left[\frac{a'^F_m}{r'|m|} + b'^F_m H_m(\lambda' r')\right]r^{-|m|} - \frac{\omega^2}{\lambda'^2}\frac{1}{2|m|}\left(\frac{r_<}{r_>}\right)^{|m|}\\ &+\left[\frac{A'^F_m}{r'|m|} + B'^F_m H_m(\lambda' r')\right]H_m(\lambda' r) - \frac{\omega^2}{\lambda'^2}\frac{\pi}{2\mathrm{i}}J_m(\lambda' r_<)H_m(\lambda' r_>),\end{aligned} \tag{18}$$

while G'_m has the same form with the constants $a'^F_m, b'^F_m, A'^F_m, B'^F_m$ replaced by $a'^G_m, b'^G_m, A'^G_m, B'^G_m$, respectively. The homogeneous differential equations have solutions

$$\tilde{F}'_m(r, r') = \frac{\omega^2}{\lambda'^2}\left[\frac{a'^{\tilde{F}}_m}{r'^{|m|}} + b'^{\tilde{F}}_m H_m(\lambda' r')\right]r^{-|m|} + \left[\frac{A'^{\tilde{F}}_m}{r'|m|} + B'^{\tilde{F}}_m H_m(\lambda' r')\right]H_m(\lambda' r), \tag{19}$$

while in $\tilde{G}'_m$ we replace $a'\tilde{F} \to a'^{\tilde{G}}$, etc.

When the source point is outside and the field point is inside, all the Green's functions satisfy the homogeneous equations (16) with $\varepsilon, \mu \neq 1$, and then $F_m, G_m, \tilde{F}_m, \tilde{G}_m$ are of the

[7] For details see [3, 9].

same form as in equation (19) with the corresponding change of constants. In all of the above, the outside and inside forms of λ are given by $\lambda'^2 = \omega^2 - k^2$ and $\lambda^2 = \omega^2\mu\varepsilon - k^2$.

The various constants are to be determined, as far as possible, by the boundary conditions at $r = a$. The boundary conditions at the surface of the dielectric cylinder are the continuity of tangential components of the electric field, of the normal component of the electric displacement, of the normal component of the magnetic induction and of the tangential components of the magnetic field (we assume that there are no surface charges or currents). In terms of the Green's dyadics, the conditions read

$$\hat{\mathbf{r}}\cdot\varepsilon\mathbf{\Gamma}'\Big|_{r=a-}^{r=a+} = \mathbf{0}, \qquad \hat{\boldsymbol{\theta}}\cdot\mathbf{\Gamma}'\Big|_{r=a-}^{r=a+} = \mathbf{0}, \qquad \hat{\mathbf{z}}\cdot\mathbf{\Gamma}'\Big|_{r=a-}^{r=a+} = \mathbf{0}, \tag{20a}$$

$$\hat{\mathbf{r}}\cdot\mu\mathbf{\Phi}\Big|_{r=a-}^{r=a+} = \mathbf{0}, \qquad \hat{\boldsymbol{\theta}}\cdot\mathbf{\Phi}\Big|_{r=a-}^{r=a+} = \mathbf{0}, \qquad \hat{\mathbf{z}}\cdot\mathbf{\Phi}\Big|_{r=a-}^{r=a+} = \mathbf{0}. \tag{20b}$$

By imposing these boundary conditions, we find that the only constants contributing to the energy are

$$B_m^{\tilde{G}} = -\frac{\varepsilon^2}{\mu}(1-\varepsilon\mu)\frac{mk\omega}{\lambda\lambda' D}J_m(\lambda a)H_m(\lambda' a)B_m^F, \tag{21a}$$

$$B_m'^{\tilde{G}} = -\left(\frac{\lambda}{\lambda'}\right)^2\frac{\varepsilon}{\mu}(1-\varepsilon\mu)\frac{mk\omega}{\lambda\lambda' D}J_m^2(\lambda a)B_m^F, \tag{21b}$$

$$B_m'^{F} = \frac{\omega^2}{\lambda'^2}\frac{\pi}{2\mathrm{i}}\frac{J_m(\lambda' a)}{H_m(\lambda' a)} + \left(\frac{\lambda}{\lambda'}\right)^2\frac{\varepsilon}{\mu}\frac{J_m(\lambda a)}{H_m(\lambda' a)}B_m^F, \tag{21c}$$

$$B_m^{\tilde{F}} = -\frac{\mu}{\varepsilon^2}(1-\varepsilon\mu)\frac{mk\omega}{\lambda\lambda'\tilde{D}}J_m(\lambda a)H_m(\lambda' a)B_m^G, \tag{21d}$$

$$B_m'^{\tilde{F}} = -\left(\frac{\lambda}{\lambda'}\right)^2\frac{1}{\varepsilon}(1-\varepsilon\mu)\frac{mk\omega}{\lambda\lambda'\tilde{D}}J_m^2(\lambda a)B_m^G, \tag{21e}$$

$$B_m'^{G} = \frac{\omega^2}{\lambda'^2}\frac{\pi}{2\mathrm{i}}\frac{J_m(\lambda' a)}{H_m(\lambda' a)} + \left(\frac{\lambda}{\lambda'}\right)^2\frac{1}{\varepsilon}\frac{J_m(\lambda a)}{H_m(\lambda' a)}B_m^G, \tag{21f}$$

all in terms of $B_m^F = -\frac{\mu}{\varepsilon}\frac{\omega^2}{\lambda\lambda'}\frac{D}{\Xi}$ and $B_m^G = -\varepsilon\frac{\omega^2}{\lambda\lambda'}\frac{\tilde{D}}{\Xi}$.

The denominators occurring here are[8]

$$\Xi = (1-\varepsilon\mu)^2\frac{m^2k^2\omega^2}{\lambda^2\lambda'^2}J_m^2(\lambda a)H_m^2(\lambda' a) - D\tilde{D}, \tag{22a}$$

$$D = \varepsilon\lambda' a J_m'(\lambda a)H_m(\lambda' a) - \lambda a H_m'(\lambda' a)J_m(\lambda a), \tag{22b}$$

$$\tilde{D} = \mu\lambda' a J_m'(\lambda a)H_m(\lambda' a) - \lambda a H_m'(\lambda' a)J_m(\lambda a). \tag{22c}$$

It is now easy to check that the terms in the Green's functions that involve powers of r or r' do not contribute to the electric or magnetic field. So, even though we are not able to determine all the constants (note that there is some ambiguity in these since they cannot be

[8] The denominator structure appearing in Ξ is precisely that given by Stratton [5] and is the basis for the calculation given for example in [7]. It is also employed in an independent rederivation of the Casimir energy for a dilute dielectric cylinder [8].

uniquely determined), it is not an issue since the energy will be well defined [3, 9]. These constants always enter in the same form and therefore their individual values are not relevant. As we might have anticipated, only the pure Bessel function terms contribute[9].

1.3. Stress on the cylinder

We are now in a position to calculate the pressure on the surface of the cylinder from the radial–radial component of the stress tensor

$$P = \langle T_{rr}\rangle(a-) - \langle T_{rr}\rangle(a+), \tag{23}$$

where $T_{rr} = \frac{1}{2}\left[\varepsilon\left(E_\theta^2 + E_z^2 - E_r^2\right) + \mu\left(H_\theta^2 + H_z^2 - H_r^2\right)\right]$. As a result of the boundary conditions, the pressure on the cylindrical walls is given by the expectation value of the squares of field components just outside the cylinder; therefore,

$$T_{rr}\big|_{a-} - T_{rr}\big|_{a+} = \frac{\varepsilon - 1}{2}\left(E_\theta^2 + E_z^2 + \frac{E_r^2}{\varepsilon}\right)\bigg|_{a+} + \frac{\mu - 1}{2}\left(H_\theta^2 + H_z^2 + \frac{H_r^2}{\mu}\right)\bigg|_{a+}, \tag{24}$$

where the expectation values are given by (6) in terms of the Green's functions. We obtain the pressure on the cylinder as

$$\begin{aligned} P = \frac{\varepsilon - 1}{16\pi^3 a^4}\sum_{m=-\infty}^{\infty}\int_{-\infty}^{\infty} \mathrm{d}\zeta a\, \mathrm{d}ka \frac{\hbar}{\tilde{\Xi}}\bigg\{ & K_m'^2(y')I_m(y)I_m'(y)y(k^2a^2 - \zeta^2a^2\mu) - K_m'(y')I_m^2(y) \\ & \times K_m(y')\left[\frac{m^2k^2a^2\zeta^2a^2}{y'^3\varepsilon}\left(-2(\varepsilon+1)(1-\varepsilon\mu) + \frac{k^2a^2 - \zeta^2a^2\varepsilon}{y^2}(1-\varepsilon\mu)^2\right)\right. \\ & \left. - \frac{y^2}{y'}\left(\frac{m^2}{y'^2}\left(k^2a^2 - \frac{\zeta^2a^2}{\varepsilon}\right) + y'^2\right)\right] - K_m'(y')I_m'^2(y)K_m(y')\mu y'(k^2a^2 - \zeta^2a^2\varepsilon) \\ & - I_m(y)I_m'(y)K_m^2(y')y\left[\frac{m^2}{y'^2}(k^2a^2\mu - \zeta^2a^2) + y'^2\mu\right]\bigg\} + \{(\varepsilon \leftrightarrow \mu)\}, \end{aligned} \tag{25}$$

where we have performed the Euclidean rotation $\omega \to \mathrm{i}\zeta$, $\lambda \to \mathrm{i}\kappa$, and $\tilde{\Xi}$ is the rotated Ξ. Here $y = \kappa a$, $y' = \kappa' a$ and the last bracket indicates that the expression there is similar to that for the electric part by switching ε and μ, showing manifest symmetry between the electric and magnetic parts. However, this expression is incomplete. It contains an unobservable 'bulk' energy contribution, which the formalism would give if a medium either that of the interior with dielectric constant ε and permeability μ or that of the exterior with dielectric constant and permeability unity fills all the space [10]. The corresponding stresses are computed from the free Green's functions which satisfy (15) and have solutions

$$F_m^{(0)}(r, r') = \frac{\mu}{\varepsilon}G_m^{(0)}(r, r') = -\frac{\omega^2\mu}{\varepsilon\lambda^2}\left[\frac{1}{2|m|}\left(\frac{r_<}{r_>}\right)^{|m|} + \frac{\pi}{2\mathrm{i}}J_m(\lambda r_<)H_m(\lambda r_>)\right], \tag{26}$$

where $0 < r, r' < \infty$. Note that in this case both $\tilde{F}_m^{(0)}$ and $\tilde{G}_m^{(0)}$ are zero. After the Euclidean rotation, the bulk pressure becomes

$$\begin{aligned} P^b = T_{rr}^{(0)}(a-) - T_{rr}^{(0)}(a+) = \frac{\hbar}{16\pi^3 a^4}\sum_{m=-\infty}^{\infty}\int_{-\infty}^{\infty}\mathrm{d}\zeta a\,\mathrm{d}ka\{ & y^2 I_m'(y)K_m'(y) \\ & - (y^2 + m^2)I_m(y)K_m(y) - y'^2 I_m'(y')K_m'(y') + (y'^2 + m^2)I_m(y')K_m(y')\}. \end{aligned} \tag{27}$$

This term must be subtracted from the pressure given in (25). Note that $P^b = 0$ in the special case $\varepsilon\mu = 1$ as it should be.

[9] It might be thought that $m = 0$ is a special case, and indeed $\frac{1}{2|m|}\left(\frac{r_<}{r_>}\right)^{|m|} \to \frac{1}{2}\ln\frac{r_<}{r_>}$, but just as the latter is correctly interpreted as the limit as $|m| \to 0$, so the coefficients in the Green's functions turn out to be just the $m = 0$ limits for those given above, so the $m = 0$ case is properly incorporated.

2. Dilute dielectric cylinder

We now turn to the case of a dilute dielectric medium filling the cylinder, that is, set $\mu = 1$ and consider $\varepsilon - 1$ as small. We can then expand the integrand in (25) and (27) in powers of $(\varepsilon - 1)$. Because the expression in (25) is already proportional to that factor, we need to only expand the integrand to first order. The *total* pressure can then be written as

$$P - P^b = \frac{\hbar}{8\pi^2 a^4}(\varepsilon - 1)^2 \sum_{m=-\infty}^{\infty} \int_0^\infty \mathrm{d}y \Bigg\{ \frac{y^4}{2} \Bigg[\frac{1}{2} K_m'^2(y) I_m'(y) I_m(y) + K_m'^2(y) I_m'^2(y) \frac{y}{4} - K_m'^2(y) I_m^2(y) \frac{y}{4} \left(1 + \frac{m^2}{y^2}\right) + K_m'(y) I_m'^2(y) K_m(y) + K_m^2(y) I_m^2(y) \frac{y}{2} \left(1 + \frac{m^2}{y^2}\right)\left(1 - \frac{m^2}{2y^2}\right) - K_m^2(y) I_m'^2(y) \frac{y}{2} \left(1 - \frac{m^2}{2y^2}\right) + K_m^2(y) I_m'(y) I_m(y) \left(1 + \frac{m^2}{2y^2}\right) \Bigg] + \frac{3y}{16}[I_m(y) K_m(y)]' \Bigg\}. \tag{28}$$

Thus, the total stress vanishes in leading order which is consistent with the interpretation of the Casimir energy as arising from the pairwise interaction of dilutely distributed molecules. Several methods to compute this integral are explained in great detail in [3, 9]. There it is shown that making use of the asymptotic expansion for the Bessel functions we can numerically evaluate the integral

$$P = \frac{(\varepsilon - 1)^2}{32\pi^2 a^4}(-0.007\,612 + 0.287\,168 + 0.024\,417 - 0.002\,371 - 0.000\,012 - 0.301\,590) = 0.000\,000, \tag{29}$$

and by introducing an exponential regulator $\mathrm{e}^{-\delta y}$ in (28) we can unambiguously separate the two divergent terms

$$P_{\mathrm{div}} = \frac{(\varepsilon - 1)^2}{32\pi^2 a^4}\left(\frac{13\pi^2}{32\delta^3} - \frac{315\pi}{8192\delta}\right). \tag{30}$$

The form of the divergences is exactly as expected [11, 12]. In particular, there is no $1/\delta^2$ divergence. How do we interpret these terms? It is perhaps easiest to imagine that δ is given in terms of a proper-time cut-off, $\delta = \tau/a$, $\tau \to 0+$. Then if we consider the energy, rather than the pressure, the divergent terms have the form $E_{\mathrm{div}} = e_3 \frac{aL}{\tau^3} + e_1 \frac{L}{a}\frac{1}{\tau}$. Here L is the (large) length of the cylinder. Thus, the leading divergence corresponds to an energy term proportional to the surface of the cylinder, and it therefore appears sensible to absorb it into a renormalized surface energy which enters into a phenomenological description of the material system. The $1/\tau$ divergence is more problematic. It is proportional to the ratio of the length to the diameter of the cylinder, so it seems likely that this would be interpretable as an energy term referring to the shape of the body. In any case, although the structure of the divergences is universal, the coefficients of these divergences depend in detail on the particular regularization scheme adopted. The nature of divergences in such Casimir calculations is still under active study [2, 13–15]. In contrast, the term proportional to $(\varepsilon - 1)^2/a^2$ is unique. The universality of the finite Casimir term makes it hard not to think it has some real significance. Thus, of course, it could not have been any other than that zero value given by the van der Waals calculations [7, 16, 17].

3. Conclusion

We have shown how the Green's dyadic formulation, modified for dielectric materials, exhibits a transparent way to calculate the Casimir energies of a dielectric–diamagnetic cylinder and showed that in the dilute case it coincides with that obtained by summing the van der Waals energies of the constituent molecules. However, the identity is not really that trivial, because both the van der Waals and the Casimir energies contain divergent contributions. This is particularly crucial when one is considering the self-stress of a single body rather than the energy of interaction of distinct bodies. It was nontrivial to show the analogue for the case of the dielectric sphere [18], and the calculation for the dielectric cylinder turned out to be extraordinarily difficult.

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